MATH2040 Linear Algebra II

Tutorial 10

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1 Examples:

Example 1

Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, find an orthogonal matrix P and a diagonal matrix D such that $P^*AP = D$.

Solution

Let $f(t) = \begin{vmatrix} 2-t & 1 & 1 \\ 1 & 2-t & 1 \\ 1 & 1 & 2-t \end{vmatrix} = -(t-1)^2(t-4)$. So there are two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$.

Since $E_{\lambda_1} = N(A-I) = \text{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$ and $E_{\lambda_2} = N(A-4I) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$. So $w_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, w_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are eigenvectors corresponding to 1,1 and 4 respectively. Then, we need to use Gram-Schmidt process to convert $\{w_1, w_2, w_3\}$ into an orthogonal set.

Since A is symmetric (and hence normal) we know that eigenvectors of A corresponding to distinct eigenvalues are orthogonal. So we only to need to use Gram-Schmidt process for eigenvectors in the same eigenspace.

Then, $u_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are orthogonal eigenvectors.

Finally, after normalization on $\{u_1, u_2, u_3\}$, we obtain $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \sqrt{\frac{2}{3}} \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}, v_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Hence, the required matrices $P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

Example 2

Let V be a finite-dimensional complex inner product space, and let u be a fixed unit vector in V. Define the Householder operator $H_u : V \to V$ by $H_u(x) = x - 2\langle x, u \rangle u$ for all $x \in V$. Prove the following results:

(a) H_u is linear.

(b) $H_u(x) = x$ if and only if x is orthogonal to u.

(c) $H_u(u) = -u$.

(d) $H_u^* = H_u$ and $H_u^2 = I$.

Solution

(a) For any $x, y \in V$, $c \in \mathbb{C}$,

$$\begin{aligned} H_u(x+cy) &= (x+cy) - 2\langle x+cy, u\rangle u \\ &= (x - 2\langle x, u\rangle u) + c(y - 2\langle y, u\rangle u) \\ &= H_u(x) + cH_u(y). \end{aligned}$$

So H_u is linear.

(b) " $\Rightarrow$ " Suppose $H_u(x) = x$. Then, $2\langle x, u\rangle u = 0$. Since u is a fixed unit vector, so $\langle x, u\rangle = 0$ and x is orthogonal to u.

" $\Leftarrow$ " Suppose x is orthogonal to u. Then, $\langle x, u\rangle = 0$ and so $2\langle x, u\rangle u = 0$. Therefore, $H_u(x) = x$.

(c) Note u is a unit vector, then $H_u(u) = u - 2\langle u, u\rangle u = u - 2u = -u$.

(d) For any $x, y \in V$,

$$\begin{aligned} \langle x, H_u^*(y)\rangle &= \langle H_u(x), y\rangle \\ &= \langle x - 2\langle x, u\rangle u, y\rangle \\ &= \langle x, y\rangle - 2\langle x, u\rangle\langle u, y\rangle \end{aligned}$$

and

$$\begin{aligned} \langle x, H_u(y)\rangle &= \langle x, y - 2\langle y, u\rangle u\rangle \\ &= \langle x, y\rangle - 2\langle x, u\rangle\langle u, y\rangle \end{aligned}$$

So $H_u^* = H_u$.

And for all $x \in V$

$$\begin{aligned} H_u^2(x) &= H_u(x - 2\langle x, u\rangle u) \\ &= H_u(x) - 2\langle x, u\rangle H_u(u) \\ &= x - 2\langle x, u\rangle u - 2\langle x, u\rangle(-u) \\ &= x \end{aligned}$$

So $H_u^2 = I$, and hence, we can conclude that H_u is a unitary operator on V

Example 3

Let T be a normal operator on a finite-dimensional complex inner product space V. Use the spectral decomposition $T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$, where T_i is the orthogonal projection of V on E_{λ_i}, to prove the following results:

(a) If $T^n = T_0$ for some n, then $T = T_0$.

(b) $T = -T^*$ if and only if every λ_i is an imaginary number.

Solution

(a) Note, by spectral decomposition, $T^n = (\lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k)^n = \sum_{i=1}^k \lambda_i^n T_i$ since $T_i T_j = \delta_{ij} T_i$. Then, we let v_j to be an eigenvector of T corresponding to λ_j, we have

$$0 = T_0(v_j) = T^n(v_j) = \sum_{i=1}^k \lambda_i^n T_i(v_j) = \lambda_j^n v_j.$$

Since v_j is a non-zero vector, so $\lambda_j = 0$ for all j and $T = \sum_{i=1}^k \lambda_i T_i = T_0$.

(b) Note each T_i is self-adjoint, so $T^* = \overline{\lambda_1}T_1 + \overline{\lambda_2}T_2 + \cdots + \overline{\lambda_k}T_k$. Then,

$$\begin{aligned} T = -T^* &\Leftrightarrow T(x) = -T^*(x) \quad \forall x \in V \\ &\Leftrightarrow \sum_{i=1}^{k} \lambda_i T_i(x) = -\sum_{i=1}^{k} \overline{\lambda_i} T_i(x) \quad \forall x \in V \\ &\Leftrightarrow \sum_{i=1}^{k} (\lambda_i + \overline{\lambda_i}) T_i(x) = 0 \quad \forall x \in V \\ &\Leftrightarrow \sum_{i=1}^{k} 2Re(\lambda_i) T_i(x) = 0 \quad \forall x \in V \\ &\Leftrightarrow Re(\lambda_i) = 0 \text{ for } 1 \le i \le k \end{aligned}$$

2 Exercises:

Question 1 (Section 6.5 Q21):

Let A and B be $n \times n$ complex matrices that are unitarily equivalent.

(a) Prove that tr(A^*A)=tr(B^*B). (Hint: tr(XY)=tr(YX) for any $n \times n$ matrices X and Y)

(b) Using (a) to prove that $\sum_{i,j=1}^{n} |A_{ij}|^2 = \sum_{i,j=1}^{n} |B_{ij}|^2$.

(c) Using (b) to determine whether $\begin{pmatrix} 1 & 2 \\ 2 & i \end{pmatrix}$ and $\begin{pmatrix} i & 4 \\ 1 & 1 \end{pmatrix}$ are unitarily equivalent or not.

Question 2 (Section 6.5 Q30):

Suppose that β and γ are ordered bases for an n-dimensional inner product space V. Prove that if Q is a unitary $n \times n$ matrix that changes γ- coordinates into β-coordinates, then β is orthonormal if and only if γ is orthonormal.

Question 3 (Section 6.6 Q6):

Let T be a normal operator on a finite-dimensional inner product space V. Prove that if T is a projection, then T is also an orthogonal projection.

Solution

Question 1

(a) Since A and B are unitarily equivalent, then there exists a unitary matrix P such that $A = P^*BP$. So

$$tr(A^*A) = tr((P^*BP)^*(P^*BP)) = tr((P^*B^*P)(P^*BP)) = tr(P^*B^*BP) = tr(B^*BPP^*) = tr(B^*B).$$

(b) Note

$$tr(A^*A) = \sum_{i=1}^{n} (A^*A)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} A^*_{ij}A_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{A_{ji}}A_{ji} = \sum_{i,j=1}^{n} |A_{ij}|^2.$$

Similarly, $tr(B^*B) = \sum_{i,j=1}^{n} |B_{ij}|^2$. Therefore, $\sum_{i,j=1}^{n} |A_{ij}|^2 = \sum_{i,j=1}^{n} |B_{ij}|^2$.

(c) Let $A = \begin{pmatrix} 1 & 2 \\ 2 & i \end{pmatrix}$ and $B = \begin{pmatrix} i & 4 \\ 1 & 1 \end{pmatrix}$, then $\sum_{i,j=1}^{2} |A_{ij}|^2 = 10$ and $\sum_{i,j=1}^{2} |B_{ij}|^2 = 19$, so A and B are not unitarily equivalent.

Question 2

We first write $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_n\}$.

On one hand, suppose β is an orthonormal ordered basis. As $Q = [I]_\gamma^\beta$, so $w_i = \sum_{j=1}^n Q_{ji} v_j$. Then,

$$\begin{aligned}
\langle w_i, w_j \rangle &= \langle \sum_{k=1}^n Q_{ki} v_k, \sum_{l=1}^n Q_{lj} v_l \rangle \\
&= \sum_{k=1}^n Q_{ki} \overline{Q_{kj}} \\
&= \delta_{ij}
\end{aligned}$$

because $\sum_{k=1}^n Q_{ki} \overline{Q_{kj}}$ is the inner product of i-th column and j-th column of the unitary matrix Q. Therefore, γ is also an orthonormal ordered basis.

On the other hand, since Q is unitary, so $Q^* = [I]_\beta^\gamma$ is also unitary. By the similar technique above, we can also show β is orthonormal given that γ is orthonormal.

Question 3

By definition, given that T is a projection, T is an orthogonal projection if $R(T)^\perp = N(T)$ and $R(T) = N(T)^\perp$. Since V is finite-dimensional, so it is sufficient to show $R(T)^\perp = N(T)$ only.

On one hand, for any $x \in R(T)^\perp$,

$$\langle T(x), T(y) \rangle = \langle x, T^* T(y) \rangle = \langle x, T(T^*(y)) \rangle = 0 \quad \forall y \in V$$

since T is normal and $x \in R(T)^\perp$. So $x \in N(T)$.

On the other hand, for any $x \in N(T)$,

$$\langle x, T(y) \rangle = \langle T^*(x), y \rangle = 0 \quad \forall y \in V$$

since T is normal and $||T^*(x)|| = ||T(x)|| = 0$ implies $T^*(x) = 0$. So $x \in R(T)^\perp$.